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TENTAMEN RELATIVISTIC QUANTUM MECHANICS

Monday 27-10-2008, 08.30-11.30

On the first sheet write your name, address and student number. Write your name on all other sheets.

This examination consists of four problems, with in total 15 parts. The 15 parts carry equal weight in determining the final result of this examination.

$\hbar = c = 1$. The standard representation of the 4×4 Dirac gamma-matrices is given by:

$$\gamma^0 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix}.$$

PROBLEM 1

A spinor field transforms under Lorentz transformations as

$$\psi'(x') = S(\Lambda)\psi(x)\,, \tag{1.1}$$

where Λ is the Lorentz transformation matrix and $x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$.

1(a) Show the Dirac equation is covariant under Lorentz transformations if

$$S^{-1}\gamma^\mu S = \Lambda^\mu{}_\nu \gamma^\nu\,. \tag{1.2}$$

1(b) Determine the transformation of $\bar\psi(x)$ under Lorentz transformations.

1(c) Show that $\bar\psi(x)\psi(x)$ is invariant under Lorentz transformations if

$$S^{-1} = \gamma^0 S^\dagger \gamma^0\,. \tag{1.3}$$

1(d) The interaction term of the photon field and the Dirac field in quantum electrodynamics,

$$\bar\psi(x)\gamma^\mu\psi(x)A_\mu(x)\,,$$

must be invariant under Lorentz transformations. How should the photon field $A_\mu(x)$ transform to achieve this invariance?

PROBLEM 2

The field $\phi(x)$, a solution of the Klein-Gordon equation, satisfies equal-time commutation relations

$$[\phi(x), \dot{\phi}(y)]_{x^0=y^0} = i\delta^3(\vec{x}-\vec{y}) \,, \quad [\phi(x), \phi(y)]_{x^0=y^0} = 0 \,.$$

where $\dot{\phi}(y) \equiv \partial\phi(y)/\partial y^0$.

2(a) What is the definition of the time-ordered product $T\big(\phi(x)\,\phi(y)\big)$?

2(b) Let

$$G(x,y) = < 0|\, T\big(\phi(x)\,\phi(y)\big)\, |0> \,.$$

Show that

$$(\partial_x^2 + m^2)\, G(x,y) = -i\delta^4(x-y)$$

if $\phi(x)$ satisfies the Klein-Gordon equation.

2(c) A function $\phi_0(x)$ satisfies the equation

$$(\partial_\mu\partial^\mu + m^2)\,\phi_0(x) = j(x) \,,$$

where for some $j(x)$. Express $\phi_0(x)$ in terms of G and j.

PROBLEM 3

Consider the theory of a scalar field $\phi(x)$, with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \tfrac{1}{2}m^2\phi(x)\phi(x). \tag{3.1}$$

The operator $\phi(x)$ can be written in the form

$$\phi(x) = \int \frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}} \big(a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\big)_{k^0=\omega_k} \,, \tag{3.2}$$

where

$$[a(k), a^\dagger(p)] = \delta^3(\vec{k}-\vec{p}) \,, \quad [a(k), a(p)] = [a^\dagger(k), a^\dagger(p)] = 0 \,. \tag{3.3}$$

3(a) What is the canonical momentum $\pi(x)$ corresponding to the coordinate $\phi(x)$?

3(b) Given that classically $\{\phi(t,\vec{x}), \pi(t,\vec{y})\}_{\rm PB} = \delta^3(\vec{x}-\vec{y})$, what is the result of the equal-time commutation relation

$$[\phi(x), \pi(y)]_{x^0=y^0} \tag{3.4}$$

for the quantum operators ϕ and π?

3(c) Use (3.2) and the commutation relations (3.3) to show that ($x^0 \neq y^0$!)

$$[\phi(x), \phi(y)] = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(e^{-ik(x-y)} - e^{+ik(x-y)}\right)_{k^0=\omega_k} . \tag{3.5}$$

3(d) Use the result (3.5) to evaluate

$$[\phi(x), \partial_{y^0}\phi(y)] .$$

and determine the limit $x^0 \to y^0$.

PROBLEM 4

Consider the annihilation of an electron-positron pair into two photons:

$$e^+ + e^- \to \gamma + \gamma ,$$

with momenta

$$e^+ : \ p_1 = (E_1, \vec{p}_1), \ e^- : \ p_2 = (E_2, \vec{p}_2), \ \text{photons}: \ k_1 = (\omega_1, \vec{k}_1), \ k_2 = (\omega_2, \vec{k}_2).$$

The process takes place in the laboratory frame:

$$\vec{p}_2 = 0 .$$

4(a) Why is $\omega_i = |\vec{k}_i|$?

4(b) Express E_1 and $\vec{p}_1$ in terms of the energies and momenta of the two photons.

4(c) The two photons appear under an angle θ in this process: $\vec{k}_1 \cdot \vec{k}_2 = \omega_1\omega_2 \cos\theta$. Express $\cos\theta$ in terms of m, ω_1, ω_2.

4(d) For which energies E_1 do we get $\cos\theta \to 1$?